

Reliability Assessment of Systems with Bivariate Dependent Stresses: A Comparative Study of Marshall-Olkin Distribution and FGM Copula with Bayesian Inference

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Abstract

This paper develops a framework for the reliability of a stress-strength model that assumes the strength variable (Y) is constrained by two dependent stresses (X_1 and X_2). In contrast with classical forms of this study, which typically assume independence of the stress variables, this one considers the realistic aspect of the dependence relationship between the two variables that, if ignored, can result in biased estimates of reliability. There are two dependence structures that we use. The Marshall–Olkin model is a formulation that includes common shock effects, and the Farlie–Gumbel–Morgenstern (FGM) copula provides analytical tractability and flexible dependence models without using the marginal distributions. These two structures are compared in this study. Both maximum likelihood estimation and Bayesian estimation are employed to acquire point and interval estimates for the model parameters. The Bayesian approach is implemented with the Metropolis–Hastings algorithm. The results of the simulation indicate that the Bayesian method tends to give more precise estimates with less bias, lower Mean Squared Error, and more efficient interval estimates, particularly when the sample size is small. Both models are adequate for describing the underlying dependence; however, the FGM copula provides more accurate results for moderate dependence systems with more efficient interval estimates. The practical utility and soundness of the suggested methodologies are also demonstrated through a real-data example. In general, the paper provides a comprehensive comparison under a unified estimation framework that analyzes dual dependence structures in the estimation of $R = P(X_1 < Y < X_2)$ and emphasizes the critical importance of valid dependence modeling in stress-strength reliability.

Keywords: Bayesian estimation; Bivariate Stress-strength reliability; Dependent stress, FGM Copula; Marshall–Olkin distribution; Metropolis–Hastings algorithm.

1. Instructions

Reliability analysis is one of the foundations of engineering and applied sciences and is essential for the safe and efficient functioning of an uncertain system. The stress-strength model is a basic model in reliability testing and analyzes the likelihood that the system strength is greater than the stress applied to it, thus preventing failures.

To provide an intuitive illustration of the stress-strength framework, Figure 1 presents a schematic

diagram depicting the probability that the system strength Y exceeds the stress variable X , i.e., the system reliability $R = P(X < Y)$. The shaded area under the curve represents this reliability, highlighting the likelihood of proper system performance under varying stress conditions.

The diagram is adapted from classical reliability concepts in Stapelberg [1], which provides a standard representation of stress-strength interference. Including this schematic helps readers visualize the probabilistic relationship between stress and strength components and enhances the interpretability of the proposed model.

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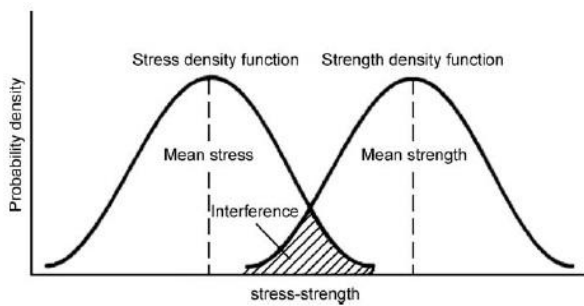


Figure 1. Stress-strength diagram. Adapted from Stapelberg [1]

There has been a broad body of research over the past decades on single-component stress-strength models, with a wide variety of data types and different distributional assumptions on both stress and strength variables. Kotz et al. [2] provided a comprehensive survey of developments up to 2003. Kundu and Gupta [3] studied single-component reliability under Weibull-distributed stress and strength, while Al-Mutairi et al. [4] considered Lindley-distributed stress and strength using both classical and Bayesian inference. De la Cruz et al. [5] analyzed independent unit half-normal distributions and constructed confidence intervals via maximum likelihood and bootstrap methods. More recently, Yousef et al. [6] derived point and interval estimators based on progressively Type-II censored samples from Burr type-XII distributions under step-stress partially accelerated life testing.

Stress and strength are generally not independent in stress-strength modeling. Their interdependence can have a significant impact on system reliability. Two major approaches have been developed to capture such dependencies. The first approach assumes that stress and strength follow a joint distribution, such as the bivariate Weibull [7], bivariate conditional exponential [8], or bivariate log-normal distribution [9]; this method typically requires both marginal distributions to belong to the same family. The second approach employs copula functions, which model dependence structures independently of the marginal distributions. Early pioneers in copula-based modeling include Nelsen [10], Joe [11], and Durante and Sempi [12]. The Farlie-Gumbel-Morgenstern (FGM) copula in stress-strength analysis with Burr III marginals was introduced by Domma and Giordano [13], and the FGM copula with Rayleigh marginals in the maximum likelihood, marginal inference, and semi-parametric methods was applied by James et al. [14]. Hakamipour [15-17] also applied copula-based approaches to multicomponent systems and s-out-of-k systems and dependent competing risks in the context of accelerated life tests.

The majority of existing research presupposes that the strength variable Y is directly compared with one stress variable X . In most real-world situations, however, Y should be between two dependent stresses, X_1 and X_2 , like devices that are not functional in some temperature limits, blood pressure with systolic and diastolic limits, or

processes with lower and upper limits. This motivates the interest in studying bounded reliability models where the most significant measure of reliability is the probability $P(X_1 < Y < X_2)$, and it has been used in manufacturing, environmental monitoring, financial risks, and process optimization.

There are a number of contributions that have been made to this bounded reliability model. Hassan et al. [18] estimated $P(X_1 < Y < X_2)$ under the Weibull distribution with outliers in the strength variable. Wang et al. [19] studied statistical inference for the bivariate stress-strength model using non-parametric normal approximation and jackknife empirical likelihood. Hameed et al. [20] proposed an estimator when Y , X_1 and X_2 are independent and follow the inverted Kumaraswamy distribution.

These studies mostly assume that stress and strength variables are independent, which is not always realistic. Stress and strength in most real-world systems are dependent on each other, and ignoring this dependence may result in biased reliability estimates. Although there have been recent advances to include dependence structures, few studies have explored the bivariate stress-strength model in the presence of dependent stresses, as this is a significant research field that has not been fully explored. One approach that is widely used and powerful is the use of copula functions, which can be flexibly used to model dependence without making assumptions about the marginals. Hudaverdi and Susam [21] investigated $P(X_1 < Y < X_2)$ under dependent stress variables using an FGM copula with Burr-III marginals. Wang and Yan [22] applied the Clayton copula in a bivariate stress-strength framework.

Although flexible, copula-based models can be computationally demanding. Parameter estimation may also be challenging, particularly in high-dimensional settings or with limited data. In many engineering applications, common shock mechanisms naturally induce dependence. These mechanisms can often be modeled more intuitively using structured multivariate distributions. Many of these coincide with the Marshall-Olkin family. When combined with the Weibull distribution, the Marshall-Olkin model provides an easily computable and physically interpretable model of dependent lifetimes. The Marshall-Olkin distribution is especially appropriate for reliability analysis and engineering because it is flexible and can model both individual and common shock effects.

Only a few studies have considered stress-strength models with dependent bivariate stresses, and most have relied predominantly on copula-based methods, without extensively exploring joint distributions. In the current study, we adopt a dual approach, employing both copula-based methods and the Marshall-Olkin distribution. The analysis focuses on their relative strengths, illustrating the flexibility of copulas in modeling dependence and the parsimonious, physically interpretable representation offered by the Marshall-Olkin model. The stress-strength

reliability measure R is investigated in the setting where component strength lies between two dependent stresses, X_1 and X_2 . To estimate the parameters, maximum likelihood estimation and Bayesian estimation are applied, with Bayesian estimation conducted using Markov chain Monte Carlo (MCMC) with Metropolis-Hastings updates modified to satisfy the parameter constraints. This comparison provides additional insight into the behavior of stress-strength systems under varying dependence levels.

The remainder of the paper is organized as follows. In Section 2, the stress-strength reliability model is introduced under bivariate stress, and R is derived for both the Marshall-Olkin and FGM copula. In Section 3, the estimation strategies are given. Section 4 reports the results of simulation studies, while Section 5 illustrates the proposed methodology using a real data example. Finally, the conclusion is presented in Section 6.

2. Reliability of The Stress-Strength Model Under Bivariate Stress

Let Y be the variable of strength, and (X_1, X_2) be the lower and upper limits of stress of a component, respectively. The system is regarded as operating successfully if the strength lies between the two stress limits, i.e., whenever $X_1 < Y < X_2$. Accordingly, the system reliability is defined as $R = P(X_1 < Y < X_2)$.

Assume that Y , X_1 , and X_2 are mutually independent random variables. Let $Y \sim \text{Weibull}(\alpha, \beta)$ have the probability density function (PDF):

$$f_Y(y) = \alpha\beta y^{\alpha-1} e^{-\beta y^\alpha}, \quad y > 0, \alpha > 0, \beta > 0. \quad (1)$$

Similarly, both stress bounds X_1 and X_2 are supposed to be Weibull distributed, with a common shape parameter α but different scale parameters λ_1 and λ_2 , i.e., $X_1 \sim \text{Weibull}(\alpha, \lambda_1)$ and $X_2 \sim \text{Weibull}(\alpha, \lambda_2)$.

Based on these assumptions, the system reliability can be expressed as

$$\begin{aligned} R &= P(X_1 < Y < X_2) \\ &= \int_0^\infty P(X_1 < y, X_2 > y \mid Y = y) dF_Y(y) \\ &= \int_0^\infty F_{X_1}(y) (1 - F_{X_2}(y)) f_Y(y) dy, \end{aligned}$$

which, after straightforward algebra, reduces

$$R = \int_0^\infty F_{X_1}(y) f_Y(y) dy - \int_0^\infty F_{X_1}(y) F_{X_2}(y) f_Y(y) dy.$$

By substituting the Weibull distribution functions, the above integrals simplify to

$$\begin{aligned} R &= \int_0^\infty (1 - e^{-\lambda_1 y^\alpha}) \alpha\beta y^{\alpha-1} e^{-\beta y^\alpha} dy \\ &\quad - \int_0^\infty (1 - e^{-\lambda_1 y^\alpha}) (1 - e^{-\lambda_2 y^\alpha}) \alpha\beta y^{\alpha-1} e^{-\beta y^\alpha} dy. \end{aligned}$$

Evaluating the integrals yields the closed-form expression:

$$R = \frac{\beta}{\lambda_2 + \beta} - \frac{\beta}{\lambda_1 + \lambda_2 + \beta}.$$

This formulation provides the baseline reliability under the independence assumption. In the subsequent

subsections, this framework will be extended to incorporate dependence between stress variables. In classical stress-strength models, independence is often assumed for analytical tractability; however, in many practical systems, stress variables are correlated, and ignoring this dependence may lead to inaccurate reliability assessments. To address this issue, we next investigate the reliability function under two dependence structures: the Marshall-Olkin bivariate distribution and the FGM copula.

2.1 Reliability with dependent stresses using the Marshall-Olkin model

Marshall and Olkin [23] introduced a bivariate Weibull-type model in which each marginal follows a Weibull distribution. The model is widely used in reliability analysis because it allows explicit modeling of dependence between component lifetimes and accounts for both individual and simultaneous shocks. In its classical formulation, two components are affected by three shock sources acting separately or jointly, enabling the representation of both individual and common failure mechanisms.

This framework has been further developed in subsequent studies. Jamalizadeh and Kundu [24] introduced a weighted extension of the Marshall-Olkin bivariate exponential model to enhance modeling flexibility. Additional theoretical results, inference procedures, and applications were provided by Kundu and Dey [25], Shoaee and Khorram [26], Abuelamayem and Aly [27], and Mukhopadhyay et al. [28]. Collectively, these contributions demonstrate the usefulness of the Marshall-Olkin family for modeling dependent lifetime data.

The distribution may be characterized through its joint survival function for stress variables X_1 and X_2 , it is given by:

$$\bar{F}_{X_1, X_2}(x_1, x_2) = e^{-\lambda_1 x_1^\alpha - \lambda_2 x_2^\alpha - \lambda_3 \max\{x_1, x_2\}^\alpha}, \quad (2)$$

with the shape parameter $\alpha > 0$ and the scale parameters, $\lambda_1, \lambda_2, \lambda_3 > 0$.

Suppose the stress variables are X_1 and X_2 , and the strength variable Y is independent of the stresses. The stress components (X_1, X_2) are dependent and jointly follow a Marshall-Olkin bivariate Weibull distribution, whose joint survival function is given in Equation (2). In this model, the dependence between X_1 and X_2 is governed by the scale parameters $\lambda_1, \lambda_2, \lambda_3$, while the shape parameter α determines the Weibull marginals. Under this dependence structure, the system reliability is expressed as

$$\begin{aligned} R &= P(X_1 < Y < X_2) \\ &= \int_0^\infty P(X_1 < y, X_2 > y \mid Y = y) dF_Y(y) \\ &= \int_0^\infty (F_{X_1}(y) - F_{X_1, X_2}(y, y)) dF_Y(y). \end{aligned}$$

Based on the Marshall-Olkin joint distribution defined in Equation (2), we obtain

$$F_{X_1}(y) = 1 - F_{X_1, X_2}(y, 0) = 1 - e^{-(\lambda_1 + \lambda_3)y^\alpha}$$

and

$$F_{X_1, X_2}(y, y) = \bar{F}_{X_1, X_2}(y, y) + F_{X_1}(y) + F_{X_2}(y) - 1$$

$$= 1 - e^{-(\lambda_1 + \lambda_3)y^\alpha} - e^{-(\lambda_2 + \lambda_3)y^\alpha} + e^{-(\lambda_1 + \lambda_2 + \lambda_3)y^\alpha}.$$

Therefore, it follows that.

$$R = \frac{\beta}{\lambda_2 + \lambda_3 + \beta} - \frac{\beta}{\lambda_1 + \lambda_2 + \lambda_3 + \beta}.$$

2.2 Reliability with dependent stresses using the FGM copula

Here, we use a copula methodology to model the dependence structure between the stress variables X_1 and X_2 , providing a flexible approach for modeling multivariate distributions without constraining the marginal distributions to the dependence structure. F_{X_1} and F_{X_2} are the marginal distribution functions of X_1 and X_2 , respectively.

The concept of copula modeling is based on the theoretical foundation created by Sklar [29]. According to it, given any d -dimensional random vector $(Z_1, Z_2, \dots, Z_d)$ with joint distribution H and marginal distribution functions $F_1, F_2, \dots, F_d$, there is a copula $C: [0, 1]^d \rightarrow [0, 1]$ such that:

$$H(z_1, z_2, \dots, z_d) = C(F_1(z_1), F_2(z_2), \dots, F_d(z_d)).$$

When the functions $F_1, F_2, \dots, F_d$, which are marginal distribution functions, are continuous, the copula C is unique; if they are discontinuous, C is uniquely determined on the range of the marginals.

On the other hand, with any copula C and any univariate distribution functions $F_1, F_2, \dots, F_d$, the above function is a valid joint distribution function whose marginals are $F_1, F_2, \dots, F_d$.

The theorem by Sklar says that the joint distribution of (X_1, X_2) may be represented as:

$$F_{X_1, X_2}(x_1, x_2) = C(F_{X_1}(x_1), F_{X_2}(x_2)),$$

where $C(\cdot)$ is a bivariate distribution function with a dependence parameter θ , which defines the dependence between X_1 and X_2 . This expression suggests that any joint distribution may be constructed by joining arbitrary marginals with a suitable copula.

There are association parameters that can be included in copula families, and they measure the dependence of variables. One measure that is often adopted is Kendall's tau (τ), which was defined in terms of the copula as:

$$\tau = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1.$$

For many copula families, the association between the copula parameter θ and τ is monotonic, allowing θ to be interpreted as a measure of dependence. The FGM copula, however, is primarily appropriate for modeling weak to moderate dependence due to its limited ability to capture strong associations and tail dependence. Within this range, its analytical simplicity and separation from

marginal distributions make it a convenient and interpretable choice for comparative stress-strength reliability analysis. Accordingly, while acknowledging its restricted dependence range, the FGM copula serves as a suitable benchmark in the present study. It is defined as

$$C(u, v) = uv[1 + \theta(1 - u)(1 - v)], \quad (3)$$

where $-1 \leq \theta \leq 1$, $u = F_{X_1}(x_1)$ and $v = F_{X_2}(x_2)$. The relation between the dependence parameter θ and Kendall's tau can be expressed as $\tau = \frac{2}{9}\theta$, providing a simple interpretation of θ as the intensity of dependence.

The copula density function, representing the joint probability density via a copula, is defined as follows:

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}. \quad (4)$$

Under this framework, the system reliability in a stress-strength setup can be stipulated as:

$$R = P(X_1 < Y < X_2)$$

$$= \int_0^\infty P(X_1 < y, X_2 > y | Y = y) dF_Y(y)$$

$$= \int_0^\infty (F_{X_1}(y) - C(F_{X_1}(y), F_{X_2}(y))) dF_Y(y).$$

Through replacing the joint distribution of the stress variables with the FGM copula in Equation (3), the reliability can be rewritten as

$$R = \int_0^\infty F_{X_1}(y) dF_Y(y) - \int_0^\infty F_{X_1}(y) F_{X_2}(y) dF_Y(y)$$

$$- \theta \int_0^\infty F_{X_1}(y) (1 - F_{X_1}(y)) F_{X_2}(y) (1 - F_{X_2}(y)) dF_Y(y).$$

Lastly, the reliability is determined in closed form by replacing the cumulative distribution functions of the variables X_1 , X_2 , and Y with the following:

$$R = \frac{\beta}{\lambda_2 + \beta} - \frac{\beta}{\lambda_1 + \lambda_2 + \beta} - \theta \left(\frac{\beta}{\lambda_1 + \lambda_2 + \beta} - \frac{\beta}{\lambda_1 + 2\lambda_2 + \beta} - \frac{\beta}{2\lambda_1 + \lambda_2 + \beta} + \frac{\beta}{2\lambda_1 + 2\lambda_2 + \beta} \right).$$

3. Estimation of $R = P(X_1 < Y < X_2)$ Under Dependent Stress

To make an inference about the measure of reliability $R = P(X_1 < Y < X_2)$, we need good estimates of the underlying model parameters. In this study, we examine two different forms of dependence for the stress variables: the Marshall-Olkin bivariate equal dependence model and the model based on the FGM copula.

Once stress dependence is modeled using the Marshall-Olkin distribution, $\theta_M = (\alpha, \beta, \lambda_1, \lambda_2, \lambda_3)$ is the parameter to be estimated. Conversely, when the FGM copula representation is used, the parameter vector becomes $\theta_C = (\alpha, \beta, \lambda_1, \lambda_2, \theta)$.

For both cases, estimation procedures are developed using two complementary approaches: the maximum likelihood method and the Bayesian approach. The maximum likelihood estimators (MLEs) provide asymptotically efficient and consistent estimates when standard regularity conditions are met. In contrast, the

Bayesian framework uses prior knowledge and provides posterior distributions, allowing a detailed quantification of uncertainty.

The rest of this section is divided into two subsections. The maximum likelihood estimation of the model parameters is given in Subsection 3.1, and the Bayesian estimation framework is discussed in Subsection 3.2.

3.1 Maximum likelihood estimation

To estimate the model parameters in the two suggested dependence structures, we use the maximum likelihood method. Assuming that $X = (X_1, X_2)$ represents the joint stress, Y is the strength variable, and $\mathcal{D} = (x_{1i}, x_{2i}, y_i); i = 1, 2, \dots, n$ is a random sample of size n drawn from the joint distribution of (X_1, X_2, Y) , the likelihood function is defined as follows:

$$L(\theta; \mathcal{D}) = \prod_{i=1}^n f_{X_1, X_2}(x_{1i}, x_{2i}) f_Y(y_i).$$

In this expression $f_{X_1, X_2}(\cdot, \cdot)$ refers to the joint density of the stress variables in the dependence model chosen and $f_Y(\cdot)$ refers to the density of Y . The explicit expression of the likelihood is based on the choice of modelling the dependence between X_1 and X_2 using the Marshall-Olkin distribution or FGM copula.

3.1.1 Bivariate Marshall-Olkin distribution

According to [22], the joint PDF of X_1 and X_2 under the bivariate Marshall-Olkin distribution is expressed as

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} \lambda_2(\lambda_1 + \lambda_3)\alpha^2 x_1^{\alpha-1} x_2^{\alpha-1} e^{-(\lambda_1 + \lambda_3)x_1 - \lambda_2 x_2} & x_1 > x_2, \\ \lambda_1(\lambda_2 + \lambda_3)\alpha^2 x_1^{\alpha-1} x_2^{\alpha-1} e^{-\lambda_1 x_1 - (\lambda_2 + \lambda_3)x_2} & x_1 < x_2, \\ \alpha\lambda_3 x^{\alpha-1} e^{-(\lambda_1 + \lambda_2 + \lambda_3)x} & x_1 = x_2. \end{cases}$$

For convenience, we define the index sets $I_1 = \{i: x_{1i} > x_{2i}\}$, $I_2 = \{i: x_{1i} < x_{2i}\}$, and $I_3 = \{i: x_{1i} = x_{2i} = x\}$. Here, we have $|I_1| = n_1$, $|I_2| = n_2$, and $|I_3| = n_3$. In this setup, $n = n_1 + n_2 + n_3$. The likelihood function then takes the form:

$$L(\theta_M; \mathcal{D}) = \prod_{i=1}^n f_Y(y_i) \prod_{i \in I_1} f_{X_1, X_2}(x_{1i}, x_{2i}) \quad (6)$$

$$\prod_{i \in I_2} f_{X_1, X_2}(x_{1i}, x_{2i}) \prod_{i \in I_3} f_{X_1, X_2}(x_{1i}, x_{2i}),$$

Using the PDFs (5) and (1) in the expression of likelihood (6), we get

$$L(\theta_M; \mathcal{D}) = \alpha^{n+2n_1+2n_2+n_3} \beta^n \lambda_1^{n_2} \lambda_2^{n_1} \lambda_3^{n_3} (\lambda_1 + \lambda_3)^{n_1} (\lambda_2 + \lambda_3)^{n_2} (\prod_{i=1}^{n_1} y_i^{\alpha-1}) (\prod_{i=1}^{n_1} x_{1i}^{\alpha-1}) (\prod_{i=1}^{n_1} x_{2i}^{\alpha-1}) (\prod_{i=1}^{n_2} x_{1i}^{\alpha-1}) (\prod_{i=1}^{n_2} x_{2i}^{\alpha-1}) x^{n_3(\alpha-1)} e^{-\beta \sum_{i=1}^n y_i^\alpha} e^{-(\lambda_1 + \lambda_3) \sum_{i=1}^{n_1} x_{1i}^\alpha} e^{-\lambda_2 \sum_{i=1}^{n_1} x_{2i}^\alpha} e^{-\lambda_1 \sum_{i=1}^{n_2} x_{1i}^\alpha} e^{-\lambda_2 \sum_{i=1}^{n_2} x_{2i}^\alpha} e^{-(\lambda_2 + \lambda_3) \sum_{i=1}^{n_2} x_{2i}^\alpha} e^{-n(\lambda_1 + \lambda_2 + \lambda_3)x^\alpha}.$$

Taking natural logarithms yields the log-likelihood function.

$$\ell(\theta_M; \mathcal{D}) = (n + 2n_1 + 2n_2 + n_3) \log \alpha + n \log \beta + n_2 \log \lambda_1 + n_1 \log \lambda_2 + n_3 \log \lambda_3 + n_1 \log(\lambda_1 + \lambda_3) + n_2 \log(\lambda_2 + \lambda_3) + n_3(\alpha - 1) \log x - (\lambda_1 +$$

$$\lambda_3) \sum_{i=1}^{n_1} x_{1i}^\alpha - \lambda_1 \sum_{i=1}^{n_2} x_{1i}^\alpha - (\lambda_2 + \lambda_3) \sum_{i=1}^{n_2} x_{2i}^\alpha - n(\lambda_1 + \lambda_2 + \lambda_3)x^\alpha - \beta \sum_{i=1}^n y_i^\alpha - \lambda_2 \sum_{i=1}^{n_1} x_{2i}^\alpha + (\alpha - 1) [\sum_{i=1}^n \log y_i + \sum_{i=1}^{n_1} \log x_{1i}] + (\alpha - 1) [\sum_{i=1}^{n_1} \log x_{2i} + \sum_{i=1}^{n_2} \log x_{1i} + \sum_{i=1}^{n_2} \log x_{2i}].$$

The unknown parameters are estimated by maximizing the log-likelihood function $\ell(\theta_M)$ with respect to each element of $\theta_M = (\alpha, \beta, \lambda_1, \lambda_2, \lambda_3)$, by setting the first-order derivatives equal to zero:

$$\frac{\partial \ell}{\partial \alpha} = n_3 \log x + \sum_{i=1}^n \log y_i + \sum_{i=1}^{n_1} \log x_{1i} + \sum_{i=1}^{n_2} \log x_{2i}$$

$$+ \sum_{i=1}^{n_1} \log x_{2i} + \sum_{i=1}^{n_2} \log x_{1i} + \frac{n+2n_1+2n_2+n_3}{\alpha} - n(\lambda_1 + \lambda_2 + \lambda_3)x^\alpha \log x - (\lambda_1 + \lambda_3) \sum_{i=1}^{n_1} x_{1i}^\alpha \log x_{1i} - \lambda_1 \sum_{i=1}^{n_2} x_{1i}^\alpha \log x_{1i} - (\lambda_2 + \lambda_3) \sum_{i=1}^{n_2} x_{2i}^\alpha \log x_{2i} - \lambda_2 \sum_{i=1}^{n_1} x_{2i}^\alpha \log x_{2i} - \beta \sum_{i=1}^n y_i^\alpha \log y_i = 0,$$

$$\frac{\partial \ell}{\partial \lambda_1} = \frac{n_2}{\lambda_1} + \frac{n_1}{\lambda_1 + \lambda_3} - \sum_{i=1}^{n_1} x_{1i}^\alpha - \sum_{i=1}^{n_2} x_{1i}^\alpha - nx^\alpha = 0,$$

$$\frac{\partial \ell}{\partial \lambda_2} = \frac{n_1}{\lambda_2} + \frac{n_2}{\lambda_2 + \lambda_3} - \sum_{i=1}^{n_1} x_{2i}^\alpha - \sum_{i=1}^{n_2} x_{2i}^\alpha - nx^\alpha = 0,$$

$$\frac{\partial \ell}{\partial \lambda_3} = \frac{n_3}{\lambda_3} + \frac{n_1}{\lambda_1 + \lambda_3} + \frac{n_2}{\lambda_2 + \lambda_3} - \sum_{i=1}^{n_1} x_{1i}^\alpha - \sum_{i=1}^{n_2} x_{2i}^\alpha - nx^\alpha = 0,$$

and

$$\hat{\beta} = \frac{n}{\sum_{i=1}^n y_i^\alpha}.$$

These equations are nonlinear, and therefore, the system is numerically solved by using the Newton-Raphson algorithm. The ensuing estimates are asymptotically efficient and consistent with a standard regularity condition, and underlie further inference about the reliability measure $R = P(X_1 < Y < X_2)$.

(5) 3.1.2 FGM copula model

We then look at estimating the model parameters when the dependence structure that exists between X_1 and X_2 is being modeled as the FGM copula. Assuming that the marginal distributions and the copula are continuous, the likelihood function of the observed data $\mathcal{D}$, is defined to be:

$$L(\theta_C; \mathcal{D}) = \prod_{i=1}^n f_Y(y_i) f_{X_1}(x_{1i}) f_{X_2}(x_{2i}) \times c(F_{X_1}(x_{1i}), F_{X_2}(x_{2i})). \quad (7)$$

Based on the FGM copula defined in (3) and Equation (4), the copula density $c(\cdot, \cdot)$ in the joint distribution is given by

$$c(F_{X_1}(x_{1i}), F_{X_2}(x_{2i})) = 1 + \theta(1 - 2F_{X_1}(x_{1i}))(1 - 2F_{X_2}(x_{2i})). \quad (8)$$

By substituting the distribution functions of X_1 and X_2 into (8) and incorporating the result into the likelihood function in (7), we obtain

$$L(\theta_C; \mathcal{D}) = \alpha^{3n} \beta^n \lambda_1^n \lambda_2^n \prod_{i=1}^n x_{1i}^{\alpha-1} x_{2i}^{\alpha-1} y_i^{\alpha-1} e^{-\beta y_i^\alpha} e^{-\lambda_1 x_{1i}^\alpha} e^{-\lambda_2 x_{2i}^\alpha} (1 + \theta(2e^{-\lambda_1 x_{1i}^\alpha} - 1)(2e^{-\lambda_2 x_{2i}^\alpha} - 1)).$$

The corresponding log-likelihood function is

$$\begin{aligned} \ell(\theta_C; \mathcal{D}) = & 3n \log \alpha + n \log \beta + n \log \lambda_1 + n \log \lambda_2 \\ & - \beta \sum_{i=1}^n y_i^\alpha - \lambda_1 \sum_{i=1}^n x_{1i}^\alpha - \lambda_2 \sum_{i=1}^n x_{2i}^\alpha \\ & + (\alpha - 1) \sum_{i=1}^n (\log y_i + \log x_{1i} + \log x_{2i}) \\ & + \sum_{i=1}^n \log (1 + \theta (2e^{-\lambda_1 x_{1i}^\alpha} - 1)(2e^{-\lambda_2 x_{2i}^\alpha} - 1)). \end{aligned}$$

The first-order derivatives of the log-likelihood function with respect to the model parameters are then computed:

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} = & \frac{3n}{\alpha} - \lambda_1 \sum_{i=1}^n x_{1i}^\alpha \log x_{1i} - \lambda_2 \sum_{i=1}^n x_{2i}^\alpha \log x_{2i} - \\ & \beta \sum_{i=1}^n y_i^\alpha \log y_i + \sum_{i=1}^n (\log y_i + \log x_{1i} + \log x_{2i}) + \\ & \sum_{i=1}^n \frac{2\theta x_{2i}^\alpha (2x_{1i}^\alpha - 1) \log x_{2i} + 2\theta x_{1i}^\alpha (2x_{2i}^\alpha - 1) \log x_{1i}}{1 + \theta (2x_{1i}^\alpha - 1)(2x_{2i}^\alpha - 1)} = 0, \\ \frac{\partial \ell}{\partial \theta} = & \sum_{i=1}^n \frac{(2x_{1i}^\alpha - 1)(2x_{2i}^\alpha - 1)}{1 + \theta (2x_{1i}^\alpha - 1)(2x_{2i}^\alpha - 1)} = 0, \end{aligned}$$

and

$$\begin{aligned} \hat{\beta} &= \frac{n}{\sum_{i=1}^n y_i^\alpha}, \\ \hat{\lambda}_1 &= \frac{n}{\sum_{i=1}^n x_{1i}^\alpha}, \\ \hat{\lambda}_2 &= \frac{n}{\sum_{i=1}^n x_{2i}^\alpha}. \end{aligned}$$

The derivatives with respect to β , λ_1 , and λ_2 can be expressed in closed forms, and the derivatives with respect to α and θ cannot be computed analytically. These derivatives were therefore assessed numerically by the Newton-Raphson method. This yields the maximum likelihood estimates of the parameters under the FGM copula model.

3.2 Bayesian inference

In this subsection, we consider the Bayesian estimation of the unknown parameters in the proposed reliability model. Unlike the maximum likelihood approach, the Bayesian framework incorporates prior information about the parameters. It provides the full posterior distribution, thereby enabling both point estimation and the derivation of credible intervals.

Let $\theta = \theta_M = \theta_C = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ denote the vector of unknown parameters describing the marginal distributions and the dependence structure. Each marginal parameter, as well as the Marshall-Olkin dependence parameter, is assigned an independent Gamma prior with shape parameter a_i and scale parameter b_i . This specification follows common practice in Bayesian reliability studies (e.g., Kundu and Gupta [29]; Abuelamayem and Aly [30]), where Gamma priors are frequently adopted for positive parameters due to their flexibility, conjugacy properties, and appropriate support.

For the FGM copula dependence parameter $\theta \in [-1, 1]$, a reparameterization is employed to ensure that the prior is defined on the admissible parameter space:

$$\theta = 2\eta - 1, \quad \eta \sim \text{Beta}(a_5, b_5).$$

This transformation yields a flexible prior capable of representing weak-to-moderate dependence while respecting the theoretical range of the parameter.

Prior independence is assumed as a weakly informative choice, reflecting the absence of prior information linking the marginal parameters to the dependence structure. Such assumptions are standard in reliability modeling, where parameters typically govern distinct physical mechanisms. Under this assumption, the joint prior distribution of θ , denoted by $\pi(\theta)$, is expressed as the product of the individual marginal priors:

$$\pi(\theta) = \prod_{i=1}^5 \pi(\theta_i).$$

Given the observed sample $\mathcal{D} = (x_{1i}, x_{2i}, y_i); i = 1, 2, \dots, n$, the joint posterior distribution of the parameters is given by

$$\pi^*(\theta | \mathcal{D}) = \frac{L(\theta; \mathcal{D}) \pi(\theta)}{\int_{\Theta} L(\theta; \mathcal{D}) \pi(\theta) d\theta}. \quad (9)$$

A closed-form expression of (9) is not available; therefore, Bayesian estimates are computed using simulation-based methods. Specifically, under the squared-error loss function, the Bayes estimator of θ_i is the posterior mean:

$$\hat{\theta}_i^{\text{Bayes}} = E(\theta_i | \mathcal{D}) = \int_{\Theta} \theta_i \pi^*(\theta | \mathcal{D}) d\theta.$$

Since the posterior distribution does not belong to a standard parametric family, MCMC techniques are employed to generate posterior samples. This approach enables both point and interval estimation. In particular, the Metropolis-Hastings (MH) algorithm, optionally combined with a Gibbs sampling step, is used to generate posterior draws. For parameters whose conditional posteriors follow Gamma distributions, direct sampling is performed; otherwise, a normal random-walk MH step is applied. Posterior computations based on these samples yield Bayes estimates, while empirical quantiles of the posterior draws provide credible intervals. This methodology is applied consistently across both dependence schemes considered in this work: the bivariate Marshall-Olkin distribution and the FGM copula model.

3.3 Metropolis-Hastings algorithm

The MH algorithm is used to obtain posterior samples of parameters whose conditional distributions are not available in closed form. Initial values are specified for all unknown parameters, including those governing the marginal distributions and the dependence structure. The MH procedure is summarized as follows:

Step 1. Indicate the number of iterations N and the burn-in period B for the sampler.

Step 2. Initialize the parameters $\theta^{(0)}$ using the MLEs. Denote the parameter values at iteration t by $\theta^{(t)}$, and those at iteration $t + 1$ by $\theta^{(t+1)}$.

Step 3. Generate a candidate θ^* from a proposal distribution. For the marginal parameters and the Marshall-Olkin dependence parameter, independent normal random-walk proposals are used: i.e., $\theta_i^* \sim$

$N(\theta_i^{(t)}, \text{Var}(\hat{\theta}_i))$. For the FGM copula dependence parameter $\theta = 2\eta - 1$, MH is performed on the transformed variable $\eta \in [0,1]$ using a Beta proposal to ensure that $\theta^* \in [-1,1]$.

Step 4. Compute the Hastings ratio

$$P_{\theta} = \frac{L(\theta^*|\mathcal{D}) \pi(\theta^*) q(\theta^{(t)}|\theta^*)}{L(\theta^{(t)}|\mathcal{D}) \pi(\theta^{(t)}) q(\theta^*|\theta^{(t)})}.$$

Let $q(\theta^{(t)} | \theta^*)$ denote the proposal distribution used to generate a candidate $\theta^{(t)}$ from the current state $\theta^{(t)}$. In this study, for symmetric random-walk proposals, we have

$$q(\theta^{(t)} | \theta^*) = q(\theta^* | \theta^{(t)}),$$

and thus, the proposal terms cancel in the Hastings ratio. Consequently,

$$\varphi(\theta^* | \theta^{(t)}) = \min\{1, P_{\theta}\},$$

where φ is the acceptance probability of θ^* .

Step 5. Generate $u \sim U(0,1)$. If $u \leq \varphi(\theta^* | \theta^{(t)})$, accept θ^* and set $\theta^{(t+1)} = \theta^*$; otherwise, retain $\theta^{(t+1)} = \theta^{(t)}$.

Step 6. Repeat Steps 3–5 for N iterations.

Step 7. Discard the first B iterations as burn-in.

Bayes estimates of the parameters under squared-error loss using the retained posterior samples are calculated as:

$$\hat{\theta} = \frac{1}{N-M} \sum_{t=M+1}^N \theta^{(t)},$$

with the corresponding mean squared error (MSE)

$$\text{MSE}(\hat{\theta}) = \frac{1}{N-M} \sum_{t=M+1}^N (\theta^{(t)} - \hat{\theta})^2.$$

On the same note, the posterior mean of the reliability $R = P(X_1 < Y < X_2)$ is derived as $\hat{R} = \frac{1}{N-M} \sum_{t=M+1}^N R^{(t)}$ using the posterior samples $R^{(t)}$.

Based on the posterior samples obtained from the MH algorithm, the $100(1-\gamma)\%$ highest posterior density (HPD) credible intervals for each model parameter θ_i , $i = 1, 2, \dots, 5$, are constructed following the approach recently proposed and applied by [10].

Let $\hat{\theta}_i^{(M+1)} < \hat{\theta}_i^{(M+2)} < \dots < \hat{\theta}_i^{(N)}$ denote the ordered posterior draws for the parameter θ_i after discarding the burn-in samples. Then, the $100(1-\gamma)\%$ HPD credible interval for θ_i is given by

$$\left(\hat{\theta}_i^{[\frac{\gamma}{2}(N-M)]}, \hat{\theta}_i^{[(1-\frac{\gamma}{2})(N-M)]} \right),$$

where $0 \leq \gamma \leq 1$, and $[x]$ represents the smallest integer greater than or equal to x .

These ranges provide a probabilistic interval for the model parameters as well as the reliability measure, representing the posterior uncertainty captured by the MCMC samples.

4. Simulation Study

A simulation study was conducted to evaluate the reliability measure $R = P(X_1 < Y < X_2)$ under dependence structures induced by the Marshall–Olkin

model and the FGM copula. The objective was to examine how varying dependence levels between the stress variables influence the reliability function.

Let X_1 and X_2 denote the stress variables and Y the strength variable. The strength variable was generated independently, while dependence between the stresses was introduced through the specified dependence models.

Dependent stress samples were generated according to each model, and the corresponding sampling algorithms are presented below.

Generating dependent stress pairs (X_1, X_2) using the Marshall–Olkin bivariate Weibull distribution:

Step 1. Generate three independent uniform random variables $U_1, U_2, U_3 \sim \text{Uniform}(0,1)$.

Step 2. Transform these variables into Weibull-distributed shock times: $T_1 = (-\log U_1)/\lambda_1^{\frac{1}{\alpha}}$, $T_2 = (-\log U_2)/\lambda_2^{\frac{1}{\alpha}}$ and $T_3 = (-\log U_3)/\lambda_3^{\frac{1}{\alpha}}$.

Step 3. Construct the dependent stresses as

$$X_1 = \min(T_1, T_3), \quad X_2 = \min(T_2, T_3).$$

Step 4. The pair (X_1, X_2) represents one realization from the Marshall–Olkin bivariate Weibull distribution.

Step 5. Repeat Steps 1–4 to obtain the desired sample size.

Generating dependent stress pairs (X_1, X_2) using the FGM copula with Weibull Marginals:

Step 1. Generate an independent uniform random variable $U_1 \sim \text{Uniform}(0,1)$.

Step 2. Generate another independent uniform variable $W \sim \text{Uniform}(0,1)$ and compute the dependent uniform variable $U_2 = U_1 - \theta U_1(1 - U_1)(1 - 2W)$.

Step 3. Apply the inverse Weibull CDF to obtain the dependent stress variables:

$$X_1 = \left(-\frac{\log(1-U_1)}{\lambda_1} \right)^{\frac{1}{\alpha}}, \quad X_2 = \left(-\frac{\log(1-U_2)}{\lambda_2} \right)^{\frac{1}{\alpha}}.$$

Step 4. The pair (X_1, X_2) represents a single FGM-dependent stress sample.

Step 5. Steps 1–4 were repeated to obtain the desired sample size.

Three sets of initial parameters were considered to allow a comprehensive assessment. The initial configurations are as follows:

Dataset 1: $\alpha = 1.5$, $\beta = 1.2$, $\lambda_1 = 0.1$, $\lambda_2 = 1.5$, with $\lambda_3 \in \{0.1, 0.5, 0.9\}$ and $\theta \in \{0.1, 0.5, 0.9\}$.

Dataset 2: $\alpha = 0.7$, $\beta = 3.5$, $\lambda_1 = 1.4$, $\lambda_2 = 0.3$, with $\lambda_3 \in \{0.1, 0.5, 0.9\}$ and $\theta \in \{0.1, 0.5, 0.9\}$.

Dataset 3: $\alpha = 1$, $\beta = 2.7$, $\lambda_1 = 2.9$, $\lambda_2 = 0.1$, with $\lambda_3 \in \{0.1, 0.5, 0.9\}$ and $\theta \in \{0.1, 0.5, 0.9\}$.

The actual reliability values were calculated analytically using the closed-form expressions derived in Section 2. Table 1 summarizes the results for all parameter combinations and dependence structures. The findings indicate that both the dependence structure and the marginal distribution parameters significantly influence the true reliability R . In all cases, R decreases as the degree of dependence between stress and strength increases, implying that a higher positive correlation leads to greater synchronization of the variables and a lower probability that Y lies between X_1 and X_2 .

The decrease in R with increasing λ_3 is more pronounced in the Marshall-Olkin model. This is because the Marshall-Olkin model captures dependence through a common shock mechanism. If one component fails due to a common shock, the probability of the other component failing shortly thereafter is high. Consequently, higher values of λ_3 strengthen this dependence and reduce the reliability estimates. In contrast, the FGM copula shows only a mild decrease in R as θ increases, as it can model only limited dependence.

Table 1. True reliability R for different dependence structures and parameter sets.

Model	Dependence Parameter	Data set	R
Independent	0	1	0.0159
	0	2	0.2480
	0	3	0.4906
Marshall-Olkin	$\lambda_3 = 0.1$	1	0.0148
	$\lambda_3 = 0.5$	1	0.0114
	$\lambda_3 = 1.0$	1	0.0090
	$\lambda_3 = 0.1$	2	0.2371
	$\lambda_3 = 0.5$	2	0.1999
	$\lambda_3 = 1.0$	2	0.1709
	$\lambda_3 = 0.1$	3	0.4655
	$\lambda_3 = 0.5$	3	0.3827
	$\lambda_3 = 1.0$	3	0.3206
FGM Copula	$\theta = 0.1$	1	0.0150
	$\theta = 0.5$	1	0.0117
	$\theta = 1.0$	1	0.0083
	$\theta = 0.1$	2	0.2466
	$\theta = 0.5$	2	0.2411
	$\theta = 1.0$	2	0.2357
	$\theta = 0.1$	3	0.4901
	$\theta = 0.5$	3	0.4883
	$\theta = 1.0$	3	0.4865

Moreover, the substantial differences in R across the three data sets demonstrate that system reliability is highly sensitive to the choice of marginal parameters.

Overall, these results highlight that accurate modeling of dependence—either through a joint distribution such as the Marshall-Olkin model or via a copula-based approach like FGM—is crucial for reliable estimation of stress-strength reliability.

We conducted a comparative analysis of the maximum likelihood and Bayesian estimations of R under varying dependence structures. Three data sets were evaluated to assess the robustness of the estimators across different levels of dependence and marginal parameter configurations. The Marshall-Olkin and FGM copula models were compared in terms of absolute bias ($|\text{Bias}|$), MSE, interval length (IL), and coverage probability (CP). Simulation results are summarized in Tables 2–7 (see Appendix).

As expected, both bias and MSE decrease with increasing sample size. For example, when $n = 200$, reductions in bias and MSE across all settings demonstrate the consistency of the proposed estimators. The mean interval length is also smaller, indicating higher estimation precision and reduced uncertainty. Coverage probabilities are close to the nominal 95%, suggesting that both asymptotic coverage intervals (ACI) and HPD intervals provide satisfactory coverage.

In all simulated scenarios, the Bayesian method tends to outperform MLE in terms of smaller bias and MSE, as well as shorter HPD intervals. This improvement is particularly evident for small sample sizes, highlighting the potential advantage of the Bayesian approach in low-data settings. In both models, under the parameter values and sample sizes considered in our simulations, higher values of the dependence parameter (λ_3 in the Marshall-Olkin model and θ in the FGM copula) tended to yield slightly lower bias and MSE, together with somewhat narrower interval estimates. These results are specific to the scenarios examined and do not imply a general theoretical guarantee. The CP values remain stable, confirming the adequacy of the proposed interval estimation procedures.

A sensitivity analysis was conducted to assess the impact of the prior distributions on the posterior estimates of the model parameters for the dependence model under the FGM copula using Data Set 1. The initial prior values were set as $a_1 = a_2 = 1.3$, $b_1 = b_2 = 1.2$, $a_3 = b_3 = 0.9$, $a_4 = b_4 = 1$, $a_5 = b_5 = 0.9$. In each step, only one prior value was varied while keeping the others fixed, and its impact on the Bayesian estimates was assessed. The results, summarized in Table 8 (see Appendix), indicate that these variations have negligible effects on the posterior summaries, suggesting that the Bayesian results are robust and confirming the suitability of the chosen priors within the considered ranges.

The comparison between the Marshall-Olkin and FGM copulas indicates that the FGM model generally yields slightly lower bias and MSE values along with shorter intervals. Although the differences are small, the FGM copula appears more efficient, likely due to its

ability to handle weak to moderate dependencies. Performance trends are similar across all data sets.

Overall, the simulation study demonstrates that both MLE and Bayesian methods provide credible results when applied to the proposed copula-based model. However, the Bayesian estimator consistently yields more accurate and efficient estimates, particularly for small sample sizes. The FGM copula exhibits slightly better performance and is therefore recommended for modeling bivariate stress-strength reliability systems with moderate dependence.

5. Real Data Example

To demonstrate the practical applicability of the proposed stress-strength reliability model, a real data set originally studied by Wang and Yan [31] is used. The data comprise three sets of observations, which have been employed in previous reliability studies to model stress-strength mechanisms and dependence structures. These data sets were reorganized and preprocessed to align with the existing framework.

Following Wang and Yan [31], the first and second data sets are treated as stress variables (X_1 and X_2), while the third serves as the strength variable (Y). All samples are standardized to 30 observations to enable efficient comparison and analysis of dependencies. Random sampling was performed in R using the `sample(.)` function to obtain maximum matching samples when necessary.

For numerical stability and computational convenience, all data values were scaled by 1000 and arranged in increasing order according to cumulative order statistics. This preprocessing ensures that the dependence between X_1 and X_2 can be examined within a unified and balanced system. The transformed data are summarized in Table 9 (see Appendix).

Table 10 (see Appendix) reports the results of Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D) tests for the real data. In both dependence structures, p-values exceed 0.05, indicating an adequate fit. The slightly higher p-values for the Marshall-Olkin model suggest a marginally better agreement with the empirical distribution. These results confirm that both models suitably capture the dependence structure and support their use for stress-strength reliability analysis.

Table 11 (see Appendix) shows a strong agreement between the MLE and Bayesian estimates of the reliability measure, confirming the consistency and stability of both approaches. The estimated values using the FGM copula are slightly higher than those obtained under the Marshall-Olkin model, suggesting that the system performs more efficiently under the weaker dependence structure modeled by the FGM copula. Furthermore, the HPD intervals from the Bayesian

analysis are narrower than the ACI, indicating higher estimation precision and lower uncertainty. Comparison with the simulation results confirms that both models adequately describe the dependence structure, while the FGM copula provides somewhat more reliable estimates and more efficient interval estimates.

6. Conclusion

This study investigates stress-strength reliability under the assumption that two dependent stress variables influence component strength. A comparative framework was considered, combining a copula-based approach with the Marshall-Olkin model, which captures dependence arising from common shock mechanisms. Reliability (R) decreases as dependence between stress variables increases, since stronger common-shock dependence leads to higher chances of joint failures. The FGM copula, while known to be limited to weak dependence structures, is analytically tractable and independent of the marginal distributions, and was shown to perform adequately within its applicable range under mild to moderate dependence.

Both MLE and Bayesian estimators were applied. Simulation results show that Bayesian estimators outperform MLEs in terms of bias, MSE, and interval precision, particularly for small sample sizes. Sensitivity analysis confirms the consistency of both approaches across varying dependence levels and parameter configurations. Ignoring dependence may result in biased reliability estimates, emphasizing the importance of correctly modeling dependence structures in stress-strength systems.

Application to a real dataset demonstrates the practicality and efficiency of the proposed methods. Goodness-of-fit tests (Kolmogorov-Smirnov and Anderson-Darling) indicate that both the Marshall-Olkin and FGM copula models adequately represent the observed data.

Overall, this paper provides a comparative contribution to reliability analysis in the presence of bivariate dependent stresses, offering insight into the functioning of the Marshall-Olkin and FGM copula models. Rather than proposing a fundamentally new modeling paradigm, the study clarifies the practical implications, advantages, and limitations of commonly used dependence structures. It supports the use of Bayesian inference in small-sample reliability problems. Careful modeling of dependence is crucial for accurate and relevant reliability assessment in complex engineering and scientific problems.

Conflict of Interest

No conflict of interest has been expressed by the authors.

7. References

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Appendix

Table 2. Estimates $\hat{R}$ under the Marshall-Olkin model for data set 1

n	λ_3		$\hat{R}$			CI	
			MLE	Bayesian		ACI	HPD
50	0.1	Bias	0.0045	0.0031	IL	0.0068	0.0062
		MSE	0.0034	0.0025	CP	0.9511	0.9544
	0.5	Bias	0.0044	0.0028	IL	0.0067	0.0059
		MSE	0.0033	0.0023	CP	0.9517	0.9540
	0.9	Bias	0.0041	0.0026	IL	0.0065	0.0057
		MSE	0.0031	0.0020	CP	0.9518	0.9538
100	0.1	Bias	0.0037	0.0022	IL	0.0053	0.0043
		MSE	0.0029	0.0019	CP	0.9507	0.9521
	0.5	Bias	0.0036	0.0020	IL	0.0050	0.0039
		MSE	0.0028	0.0017	CP	0.9491	0.9513
	0.9	Bias	0.0031	0.0018	IL	0.0046	0.0036
		MSE	0.0025	0.0016	CP	0.9497	0.9510
200	0.1	Bias	0.0025	0.0017	IL	0.0046	0.0034
		MSE	0.0021	0.0015	CP	0.9489	0.9509
	0.5	Bias	0.0028	0.0019	IL	0.0047	0.0036
		MSE	0.0023	0.0016	CP	0.9471	0.9507
	0.9	Bias	0.0023	0.0014	IL	0.0037	0.0031
		MSE	0.0019	0.0012	CP	0.9492	0.9506

Table 3. Estimates $\hat{R}$ under the Marshall-Olkin model for data set 2

n	λ_3		$\hat{R}$			CI	
			MLE	Bayesian		ACI	HPD
50	0.1	Bias	0.0046	0.0033	IL	0.0069	0.0064
		MSE	0.0035	0.0026	CP	0.9520	0.9548
	0.5	Bias	0.0042	0.0029	IL	0.0063	0.0055
		MSE	0.0032	0.0023	CP	0.9517	0.9533
	0.9	Bias	0.0044	0.0032	IL	0.0065	0.0058
		MSE	0.0033	0.0025	CP	0.9513	0.9518
100	0.1	Bias	0.0036	0.0025	IL	0.0051	0.0041
		MSE	0.0028	0.0016	CP	0.9511	0.9518
	0.5	Bias	0.0033	0.0021	IL	0.0047	0.0038
		MSE	0.0026	0.0018	CP	0.9512	0.9519
	0.9	Bias	0.0037	0.0026	IL	0.0052	0.0043
		MSE	0.0029	0.0021	CP	0.9513	0.9522
200	0.1	Bias	0.0029	0.0027	IL	0.0048	0.0044
		MSE	0.0024	0.0022	CP	0.9495	0.9497
	0.5	Bias	0.0025	0.0023	IL	0.0040	0.0039
		MSE	0.0021	0.0020	CP	0.9476	0.9469
	0.9	Bias	0.0022	0.0021	IL	0.0035	0.0034
		MSE	0.0019	0.0018	CP	0.9478	0.9482

Table 4. Estimates $\hat{R}$ under the Marshall-Olkin model for data set 3

n	λ_3		$\hat{R}$			CI	
			MLE	Bayesian		ACI	HPD
50	0.1	Bias	0.0043	0.0032	IL	0.0064	0.0052
		MSE	0.0033	0.0025	CP	0.9527	0.9539
	0.5	Bias	0.0048	0.0039	IL	0.0069	0.0058
		MSE	0.0036	0.0031	CP	0.9521	0.9535
	0.9	Bias	0.0049	0.0040	IL	0.0072	0.0065
		MSE	0.0037	0.0030	CP	0.9511	0.9527
100	0.1	Bias	0.0035	0.0027	IL	0.0048	0.0037
		MSE	0.0027	0.0022	CP	0.9518	0.9521
	0.5	Bias	0.0039	0.0030	IL	0.0055	0.0046
		MSE	0.0030	0.0024	CP	0.9514	0.9527
	0.9	Bias	0.0038	0.0028	IL	0.0053	0.0045
		MSE	0.0029	0.0023	CP	0.9513	0.9520
200	0.1	Bias	0.0025	0.0023	IL	0.0039	0.0037
		MSE	0.0021	0.0019	CP	0.9509	0.9510
	0.5	Bias	0.0031	0.0028	IL	0.0051	0.0049
		MSE	0.0024	0.0023	CP	0.9488	0.9504
	0.9	Bias	0.0028	0.0027	IL	0.0047	0.0044
		MSE	0.0022	0.0021	CP	0.9493	0.9505

Table 5: Estimates $\hat{R}$ under the FGM copula model for data set 1

n	λ_3		$\hat{R}$			CI	
			MLE	Bayesian		ACI	HPD
50	0.1	Bias	0.0039	0.0023	IL	0.0063	0.0055
		MSE	0.0029	0.0018	CP	0.9529	0.9547
	0.5	Bias	0.0044	0.0035	IL	0.0069	0.0063
		MSE	0.0033	0.0025	CP	0.9527	0.9535
	0.9	Bias	0.0042	0.0026	IL	0.0064	0.0056
		MSE	0.0032	0.0019	CP	0.9525	0.9543
100	0.1	Bias	0.0030	0.0016	IL	0.0044	0.0033
		MSE	0.0023	0.0012	CP	0.9496	0.9515
	0.5	Bias	0.0034	0.0019	IL	0.0048	0.0037
		MSE	0.0026	0.0014	CP	0.9493	0.9511
	0.9	Bias	0.0035	0.0020	IL	0.0049	0.0039
		MSE	0.0026	0.0015	CP	0.9486	0.9509
200	0.1	Bias	0.0024	0.0016	IL	0.0039	0.0033
		MSE	0.0018	0.0012	CP	0.9510	0.9512
	0.5	Bias	0.0026	0.0018	IL	0.0043	0.0036
		MSE	0.0020	0.0013	CP	0.9489	0.9510
	0.9	Bias	0.0027	0.0019	IL	0.0045	0.0037
		MSE	0.0021	0.0014	CP	0.9484	0.9507

Table 6. Estimates $\hat{R}$ under the FGM copula model for data set 2

n	λ_3		$\hat{R}$			CI	
			MLE	Bayesian		ACI	HPD
50	0.1	Bias	0.0043	0.0028	IL	0.0064	0.0054
		MSE	0.0033	0.0022	CP	0.9507	0.9532
	0.5	Bias	0.0041	0.0028	IL	0.0062	0.0054
		MSE	0.0031	0.0021	CP	0.9510	0.9530
	0.9	Bias	0.0047	0.0035	IL	0.0071	0.0066
		MSE	0.0031	0.0027	CP	0.9508	0.9529
100	0.1	Bias	0.0038	0.0038	IL	0.0053	0.0046
		MSE	0.0029	0.0027	CP	0.9496	0.9499
	0.5	Bias	0.0036	0.0025	IL	0.0051	0.0042
		MSE	0.0027	0.0019	CP	0.9511	0.9520
	0.9	Bias	0.0032	0.0019	IL	0.0046	0.0036
		MSE	0.0024	0.0014	CP	0.9515	0.9524
200	0.1	Bias	0.0021	0.0019	IL	0.0032	0.0030
		MSE	0.0016	0.0014	CP	0.9495	0.9496
	0.5	Bias	0.0024	0.0021	IL	0.0039	0.0036
		MSE	0.0018	0.0016	CP	0.9475	0.9479
	0.9	Bias	0.0028	0.0026	IL	0.0047	0.0043
		MSE	0.0021	0.0020	CP	0.9462	0.9467

Table 7. Estimates $\hat{R}$ under the FGM copula model for data set 3

n	λ_3		$\hat{R}$			CI	
			MLE	Bayesian		ACI	HPD
50	0.1	Bias	0.0041	0.0032	IL	0.0062	0.0051
		MSE	0.0031	0.0024	CP	0.9518	0.9534
	0.5	Bias	0.0035	0.0029	IL	0.0069	0.0057
		MSE	0.0035	0.0029	CP	0.9519	0.9532
	0.9	Bias	0.0046	0.0038	IL	0.0068	0.0058
		MSE	0.0035	0.0029	CP	0.9517	0.9531
100	0.1	Bias	0.0037	0.0028	IL	0.0051	0.0043
		MSE	0.0028	0.0021	CP	0.9509	0.9519
	0.5	Bias	0.0036	0.0026	IL	0.0050	0.0042
		MSE	0.0027	0.0019	CP	0.9519	0.9524
	0.9	Bias	0.0040	0.0029	IL	0.0058	0.0045
		MSE	0.0030	0.0022	CP	0.9495	0.9517
200	0.1	Bias	0.0027	0.0025	IL	0.0044	0.0039
		MSE	0.0020	0.0019	CP	0.9501	0.9507
	0.5	Bias	0.0033	0.0029	IL	0.0056	0.0052
		MSE	0.0025	0.0022	CP	0.9505	0.9508
	0.9	Bias	0.0023	0.0022	IL	0.0038	0.0035
		MSE	0.0017	0.0016	CP	0.9503	0.9510

Table 8. Sensitivity analysis of Bayesian estimates for the FGM copula model under alternative prior specifications (Data Set 1)

Parameter	Prior						Prior					
	a_1	0.9	1.1	1.3	1.5	1.7	b_1	0.9	1.1	1.3	1.5	
α		1.5028	1.5031	1.5033	1.5034	1.5036		1.5029	1.5030	1.5032	1.5035	
β		1.2019	1.2021	1.2023	1.2026	1.2027		1.2018	1.2020	1.2022	1.2025	
λ_1		0.1012	0.1013	0.1014	0.1014	0.1016		0.1012	0.1012	0.1015	0.1015	
λ_2		1.5024	1.5026	1.5028	1.5029	1.5031		1.5023	1.5026	1.5029	1.5030	
θ		0.6010	0.6011	0.6013	0.6014	0.6015		0.6010	0.6012	0.6013	0.6014	
	a_2	0.9	1.1	1.3	1.5	1.7	b_2	0.9	1.1	1.3	1.5	
α		1.5027	1.5031	1.5032	1.5033	1.5035		1.5028	1.5030	1.5033	1.5033	
β		1.2018	1.2021	1.2024	1.2025	1.2029		1.2019	1.2021	1.2023	1.2025	
λ_1		0.1012	0.1014	0.1014	0.1015	0.1016		0.1013	0.1014	0.1015	0.1017	
λ_2		1.5025	1.5027	1.5029	1.5030	1.5032		1.5024	1.5026	1.5028	1.5031	
θ		0.6010	0.6010	0.6013	0.6013	0.6014		0.6010	0.6011	0.6012	0.6012	
	a_3	0.5	0.7	0.9	1.1	1.3	b_3	0.5	0.7	0.9	1.1	
α		1.5029	1.5032	1.5033	1.5034	1.5035		1.5027	1.5030	1.5031	1.5032	
β		1.2019	1.2022	1.2023	1.2025	1.2028		1.2018	1.2019	1.2021	1.2024	
λ_1		0.1013	0.1014	0.1015	0.1017	0.1016		0.1012	0.1013	0.1014	0.1016	
λ_2		1.5026	1.5026	1.5030	1.5031	1.5033		1.5025	1.5027	1.5029	1.5030	
θ		0.6009	0.6010	0.6011	0.6012	0.6013		0.6010	0.6011	0.6012	0.6013	
	a_4	0.7	0.9	1.1	1.3	1.5	b_4	0.7	0.9	1.1	1.3	
α		1.5028	1.5032	1.5032	1.5035	1.5037		1.5028	1.5031	1.5032	1.5035	
β		1.2018	1.2021	1.2022	1.2024	1.2027		1.2019	1.2022	1.2022	1.2023	
λ_1		0.1011	0.1013	0.1014	0.1014	0.1015		0.1011	0.1012	0.1014	0.1015	
λ_2		1.5026	1.5027	1.5029	1.5029	1.5030		1.5025	1.5026	1.5030	1.5030	
θ		0.6010	0.6011	0.6012	0.6014	0.6015		0.6010	0.6011	0.6012	0.6013	
	a_5	0.5	0.7	0.9	1.1	1.2	b_5	0.5	0.7	0.9	1.1	
α		1.5030	1.5034	1.5034	1.5035	1.5035		1.5029	1.5032	1.5031	1.5032	
β		1.2019	1.2020	1.2021	1.2023	1.2026		1.2019	1.2021	1.2023	1.2025	
λ_1		0.1012	0.1013	0.1013	0.1014	0.1015		0.1013	0.1013	0.1014	0.1015	
λ_2		1.5024	1.5026	1.5028	1.5030	1.5031		1.5025	1.5027	1.5029	1.5030	
θ		0.6010	0.6011	0.6011	0.6012	0.6013		0.6011	0.6012	0.6013	0.6013	

Table 9: Real data set

Variable	Data							
Y	0.03675	0.04558	0.04801	0.07146	0.08355	0.09972	0.11385	0.11699
	0.11986	0.14596	0.16649	0.18713	0.18785	0.20016	0.24453	0.28464
	0.35070	0.37581	0.41902	0.45660	0.54744	0.57862	0.58160	0.58557
	0.59429	0.66266	0.68816	0.70736	0.75670	0.76514		
X_1	0.00700	0.11200	0.15400	0.17300	0.04528	0.17600	0.27300	1.14600
	0.29700	0.41700	0.01042	1.41700	0.58300	0.01000	0.16500	0.14000
	0.01448	0.59400	0.01600	0.02270	0.14000	0.52300	0.05362	0.04940
	0.24100	0.14600	0.13300	0.04200	0.27700	0.21800		
X_2	0.50660	0.05016	0.21213	0.69373	0.17725	0.10115	0.15148	0.35324
	0.04393	0.14138	0.25744	0.16340	0.10894	0.30390	0.53055	0.32383
	0.70074	0.42211	0.70466	0.12306	0.26290	0.67149	0.59048	0.77817
	0.72723	0.38343	0.63766	0.29127	0.18316	0.37642		

Table 10. Results of the K-S and A-D goodness-of-fit tests for the real data set.

Model	Variable	K-S	P-value	A-D	P-value
Marshall-Olkin	(X_1, X_2)	0.1217	0.6934	0.2795	0.6317
	Y	0.1125	0.7214	0.2568	0.6835
FGM Copula	(X_1, X_2)	0.2018	0.5217	0.2731	0.5429
	Y	0.1425	0.6613	0.1536	0.6845

Table 11. Estimated values of $\hat{R}$ for the real data set.

Method	Marshall-Olkin	FGM Copula
MLE	0.1835	0.2127
Bayesian	0.1812	0.2104
ACI	(0.1813,0.1858)	(0.2106,0.2148)
HPD	(0.1797,0.1828)	(0.2092,0.2120)