

A Discussion of Statistical Moment Order Selection for Vibration Signal Analysis

Fuqing Yuan 

Department of Technology and Safety, University of Tromsø, Tromsø, Norway

* fuqing.yuan@uit.no

Abstract

Reliability The use of statistical moment values as indices for bearing failure diagnosis is prevalent when faults are in their incipient stages. Assuming the defect-free signal as pure white noise, a multitude of moment orders can be used to differentiate the white noise from a signal containing defect information. This paper examines whether any specific moment order can outperform the others in general, as claimed in some state-of-the-art reports. The exact statistical distribution of a signal mixed with white noise is obtained in this paper, and we use it to derive the exact moment to determine an optimal moment order. Simulation studies are conducted to verify the methods and compare the performance of different orders.

Keywords: Moment order; Vibration; Optimal order; Bearing; Defect.

1. Introduction and Background

For bearing defect in early stage, when the frequency of the defect signature is not significant, the statistical moments such as the Skewness, Kurtosis of the signal can be used for fault diagnosis.[1-6]. The underlying principle assumes that when there is no defect in the bearing, the vibration signal is dominated by white noise, and the white noise is purely random following the Normal distribution. When the fault initializes and develops, the frequency signal induced by the defect will emerge, and it mixes with the white noise. The observed signal will then deviate from the normally distributed white noise. It is well known that white noise holds some unique statistical properties. A well-known one is that its Kurtosis value is 3[7]. The Kurtosis is essentially the 4th central moment order of the normal distribution. Essentially, for defect detection, many other orders of moments can serve the same purpose. For a normal distribution, the p^{th} order expectation is [8]:

$$E((X - \mu)^p) = \begin{cases} 0; & p \text{ is odd} \\ \sigma^p(p-1)!!; & p \text{ is even} \end{cases} \quad (1)$$

The $(p-1)!!$ is double factorial defined as $(p-1)!! = (p-1)(p-3)(p-5) \dots$. Normalizing the Eq (1) by p^{th} order standard deviation σ^p , the Eq (1) becomes

$$E\left(\left(\frac{X-\mu}{\sigma}\right)^p\right) = \begin{cases} 0; & p \text{ is odd} \\ (p-1)!!; & p \text{ is even} \end{cases} \quad (2)$$

The first 10 order values are tabulated in Table 1 as follows

Table 1. Normalized High Order Expectation Values

Order p	$E\left(\left(\frac{X-\mu}{\sigma}\right)^p\right)$
1	0
2	1
3	0
4	3
5	0
6	15
7	0
8	105
9	0
10	945

Obviously, the Kurtosis is just one of the moment orders. We can further generalize the integer p to any positive real-valued order p , i.e. $p > 0$, the absolute central moment is [8]

$$E(|X - \mu|^p) = \sigma^p \cdot \frac{2^{\frac{p}{2}} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \quad (3)$$

Normalizing the (1) by its standard deviation σ^p , similar to Eq (2), we obtain a scalar value depending only on the order p .

$$E\left(\frac{|X-\mu|^p}{\sigma^p}\right) = \frac{2^{\frac{p}{2}} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \quad (4)$$

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Given an order p , either for Eq (2) or (4), if the values significantly deviate from that of Normal distribution, it considers a violation of the white noise assumption and the observed signal contains defect information, suggesting the bearing might subject to defect. All p can serve this purpose. The $p = 4$ in Eq (2) or (4) is the most popular. Other p for example $p = 2, 3$ or others are less popular and sometimes considered as new statistical index and they are claimed with better performance [1, 2, 9]. This paper is to investigate if the optimal p exists. Related to the similar topic, recently Antoni et al. (2019, 2024) proposed methods considering the health state and false state follows a Gaussian or non-Gaussian distribution, and a likelihood ratio test-based hypothesis test was proposed [10, 11]. This paper, however, addresses the optimal order problem by assuming the health state distribution follows only the Gaussian, i.e. Normal distribution. The statistical approach involved is simpler but still efficient as the hypothesis to be established can be obtained from the simpler central limit theorem instead of using other more complex approaches.

2. A Hypothesis Test

Let the observed signal as $m(t)$ which contains white noise $w(t)$ plus the pure defect signal $s(t)$:

$$m(t) = s(t) + w(t) \quad (5)$$

The white noise is assumed to follow a normal distribution $w(t) \sim N(\mu, \sigma^2)$, where the μ and σ^2 are mean and variance as conventionally denoted. We assume the observed signal is zero-mean. Practically, the zero mean is a result of signal pre-processing such as detrend processing. Other assumptions are:

- No other signal in the observed signal other than the white noise and defect signal.

The observed $m(t)$ is of a discrete series form as

$$m_1, m_2, m_3, \dots, m_n \quad (6)$$

The white noise $w(t)$ is a series as

$$w_1, w_2, w_3, \dots, w_n \quad (7)$$

For distinct denotation, the mean and standard deviation of the observed signal $m(t)$ and white noise $w(t)$ are denoted with subscript m and w respectively. The null hypothesis supposes the observed signal $m_i, i = 1, 2, 3 \dots n$ contains no defect signal and only white noise following Normal distribution.

For white noise, according to Eq (3) and since $\mu_w = 0$, the

$$\frac{\sum |w_i|^p}{n} \rightarrow E(|w|^p) = \sigma_w^p \cdot \frac{2^{\frac{p}{2}} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \quad (8)$$

The variance of $|w_i|^p$ is as follows by using Eq (4),

$$\text{var}\left(\frac{\sum |w_i|^p}{n}\right) = \sigma_w^{2p} \cdot \frac{2^p \Gamma(\frac{2p+1}{2})}{\sqrt{\pi}} - \frac{2^p \Gamma^2(\frac{p+1}{2})}{\pi} \quad (9)$$

since

$$\text{var}(|w_i|^p) = \sigma_w^{2p} \cdot \frac{2^p \Gamma(\frac{2p+1}{2})}{\sqrt{\pi}} - \sigma_w^{2p} \cdot \frac{2^p \Gamma^2(\frac{p+1}{2})}{\pi} \quad (10)$$

For the empirical moment value defined as the $\frac{\sum |w_i|^p}{n}$, supposing w_i are independent from each other, the mean is $\sigma_w^p \cdot \frac{2^{\frac{p}{2}} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}}$ by using Eq (8) and variance is Eq (9). According to the central limit theorem, the statistics $\frac{\sum |w_i|^p}{n}$ can be approximated as Normal distribution as follows [12]:

$$N(0,1) \sim \sqrt{n} \cdot \frac{\frac{\sum |w_i|^p}{n \sigma_w^p} - \frac{2^{\frac{p}{2}} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}}}{\sqrt{\frac{2^p \Gamma(\frac{2p+1}{2})}{\sqrt{\pi}} - \frac{2^p \Gamma^2(\frac{p+1}{2})}{\pi}}} \quad (11)$$

When the null hypothesis holds, i.e. when the observed signal contains no defect information, the observed signal is thus also the white noise. Substitute the white noise in (11) by the observed signal m_i and let the σ_w approximated by σ_m , a new statistic is established as:

$$N(0,1) \sim \sqrt{n} \cdot \frac{\frac{\sum |m_i|^p}{n \sigma_m^p} - \frac{2^{\frac{p}{2}} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}}}{\sqrt{\frac{2^p \Gamma(\frac{2p+1}{2})}{\sqrt{\pi}} - \frac{2^p \Gamma^2(\frac{p+1}{2})}{\pi}}} \quad (12)$$

where the variance σ_m^2

$$\sigma_m^2 = \sum \frac{(m_i - \bar{m})^2}{n-1} = \sum \frac{(m_i)^2}{n-1} \quad (13)$$

The alternative hypothesis is the signal is not of normal distribution, corresponding to the assumption that the defect signal is immersed with white noise. Following the routine of classical hypothesis test, if the quantity (12) locates outside of the region at given significant level, unnormal event occurs, the alternative hypothesis is then accepted, suggesting a defect existing in the bearing; otherwise accepting the null hypothesis, implying no defect detected. The test only requires the observed m_i , which is advantageous, as it is no need to separate white noise from the observed signal that is challenging when defect signal is not strong enough.

3. Distribution of Signal with White Noise

The exact distribution of noise-contaminated signal can be obtained to check if the test using moment value can stand. Suppose the major signal carrying defect information is of form:

$$s(t) = a \sin(\omega t + \phi) \quad (14)$$

where a is amplitude, ω is frequency and the ϕ is phase. The above formula can be simplified as follows by scaling or variable transforming as follows.

$$s(t) = a \sin t \quad (15)$$

The $a \sin t$ is the simplest wave form that does not lose the generality for the problem for which this paper is addressing. We first consider the signal as a random value in this case, despite it is essentially deterministic. The signal length L is given very large even infinite. The expectation and variance of the amplitude can be obtained from the integral as:

$$\begin{cases} \int_0^L a \sin t \, dt / L \approx 0 \\ \int_0^L (a \sin t)^2 \, dt / L \approx \frac{a^2}{2} \end{cases} \quad (16)$$

The pure defect signal is added by the white noise, i.e. $m(t) = s(t) + w(t)$. The mean of $m(t)$ is supposed to be 0 and the variance is then

$$\sigma_m = \sqrt{\text{var}(s) + \text{var}(w)} = \sqrt{\frac{a^2}{2} + \sigma_w^2} \quad (17)$$

The $\frac{a^2}{2}$ is essentially the power of the sinusoid signal and the σ_w^2 is the power of the noise signal. The best should be a p maximizing the Eq (12), implying the largest distance between the white noise Normal distribution from signal of defect. The

$$\frac{\sum |m_i|^p}{n} \rightarrow \int_0^L |a \sin t + w(t)|^p \, dt / L \quad (18)$$

when the time L is adequately long. Unfortunately, the $\int_0^L (a \sin t + w(t))^p \, dt / L$ has no analytical solution.

There is an alternative method that can solve the question in a better way. The property of

$$m(t) = a \sin t + w(t) \quad (19)$$

has been well investigated in telecommunication. The probability density function of (19) is [13]

$$p(m) = \frac{1}{\pi} \int_0^\infty J_0(a\xi) \exp\left(-\frac{\sigma_w^2 \xi^2}{2}\right) \cos(m\xi) d\xi \quad (20)$$

where the $J_0(a\xi)$ corresponds to Bessel function of the first kind of order zero.

$$J_0(a\xi) = \sum_{k=0}^\infty \frac{(-1)^k}{(k!)^2} \left(\frac{a\xi}{2}\right)^{2k} \quad (21)$$

The PDF is symmetrical as it is obviously $p(m) = p(-m)$ given by Eq (20). The PDF can be rewritten in a series expansion form, according to the (14) on page 15 in reference [14], the

$$\begin{aligned} \int_0^\infty \xi^{2k} \exp\left(-\frac{\sigma^2 \xi^2}{2}\right) \cos(m\xi) d\xi = \\ \frac{(-1)^k \sqrt{\pi}}{2^{k+1} (\sigma/\sqrt{\pi})^{2k+1}} \exp\left(-\frac{m^2}{2\sigma^2}\right) \cdot He_{2k}\left(\frac{m}{\sigma}\right) \end{aligned} \quad (22)$$

The He is Hermite polynomial function that is of complex form. The PDF can be rearranged by substituting the Eq (22) and (21) into Eq (20), as follows:

$$p(m) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{m^2}{2\sigma^2}\right) \cdot \sum_{k=0}^\infty \frac{1}{k!^2} \cdot \left(\frac{a}{\sigma}\right)^{2k} \cdot \frac{1}{2^{2k}} \cdot He_{2k}\left(\frac{m}{\sigma}\right) \quad (23)$$

The first part of the right hand Eq (23) is the PDF of normal distribution

$$f(m) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{m^2}{2\sigma^2}\right) \quad (24)$$

Expanding the second part,

$$\begin{aligned} \sum_{k=0}^\infty \frac{1}{k!^2} \cdot (a/\sigma)^{2k} \cdot \frac{1}{2^{2k}} \cdot He_{2k}\left(\frac{m}{\sigma}\right) = 1 + (a/\sigma)^2 \cdot \frac{1}{2^2} \cdot He_2\left(\frac{m}{\sigma}\right) + \frac{1}{2!^2} \cdot (a/\sigma)^4 \cdot \frac{1}{2^4} \cdot He_4\left(\frac{m}{\sigma}\right) + \dots \end{aligned} \quad (25)$$

when $a/\sigma \gg 1$, the

$$\sum_{k=0}^\infty \frac{1}{k!^2} \cdot (a/\sigma)^{2k} \cdot \frac{1}{2^{2k}} \cdot He_{2k}\left(\frac{m}{\sigma}\right) \approx 1 \quad (26)$$

When the higher order part is ignorable, the $p(m)$ is then the Normal distribution. The second part distinct the

Signal+White noise distribution differs with the white noise. Therefore, this Eq (23) theoretically proves the assumed principle stated in the beginning has its ground. It also tells when the a/σ is small, which corresponds to the SNR, meaning a weak signal, the distribution will degenerate to the Normal distribution. Figure 1 shows the PDFs of the noise-contaminated signal distribution, where we can see when the SN ratio equals or less than 1, the distribution is very similar to that of Normal distribution. When the SN ratio is bigger than 1, the shape changes and it tends to have two symmetric peaks, significantly different from Normal distribution. From this plot, we can infer the performance of using the moment by assuming the normality could work for high SN ratio but might not be appropriate for low SN ratio since it is not able to distinct from the Normal distribution.

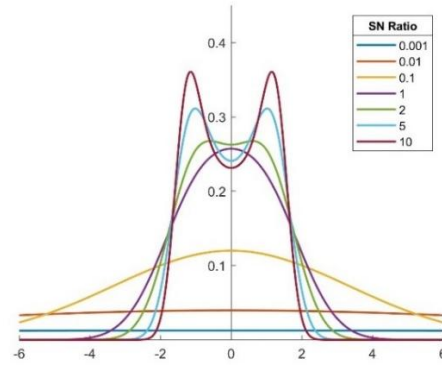


Figure 1. PDFs of Signal plus noise

For symmetric PDF, the value odd moment order is 0. For the even moment, the value is calculated from

$$Mom^k = \int_{-\infty}^{\infty} m^k p(m) dm \quad (27)$$

Numerical method is required to acquire the solution. Some simplified approaches can take advantage of the Hermite polynomial to simplify the numerical method. This paper does not cover it as it is not the main focus. Table 2 shows some example values of certain signals with specified SNR. The “%” is the relative error deviating from the expectation value of Normal distribution, defined as follows.

$$RE\% = \left| \frac{\text{Real Value} - \text{Expected}}{\text{Expected}} \right| \quad (28)$$

The “Real value” in Eq (28) is the normalized moment of (27) defined as Mom^k/σ^k . The “Expected” is the normalized moment value from a normal distribution. The higher RE% is preferable when defect is present in the machine. The “Hypothesis Value” in Table 2 is the value calculated from Eq (12) by ignoring the entity $\sqrt{n}$ and letting $\frac{\sum |m_i|^p}{n\sigma_m^p} \sim Mom^p/\sigma^p$ to avoid the selection of size n . It would not alternate the RE% value since the $\sqrt{n}$ is cancelled in Eq (28). As shown in Table 2, for all the SNR, the higher order is more sensitive than the lower ones, as indicated by the RE% where higher order with higher RE% value. But use of RE% has a clear disadvantage: for a higher order, the RE% are more sensitive to uncertainty that leads to wrong decisions. Therefore, we use the proposed hypothesis

statistic of Eq (12) to overcome the impact of uncertainty. For the order selection issue, higher hypothesis should be preferred as higher values deviate more from the Normal distribution. The results in Table 2 show that higher SNRs have higher hypothesis values for a given moment order. This is as expected as shown in Figure 1 as higher SNR implies a larger deviation from Normal distribution. However, for any given signal, hypothesis values don't vary notably. No moment order shows significantly higher value than the other, which means the moment order does not matter. The moment order selection is thus not necessary.

Table 2. Indicators of some typical signal

Moment Order		$p = 4$	$p = 6$	$p = 8$
Normalized Moment Values of Pure White Noise Normal Distribution		3	15	105
SNR=0.01 Signal 1, Noise 10	Normalized M	2.999	14.99	104.96
	RE%	0.033	0.0667	0.038
	Hypothesis Value	1e-4	9.9e-5	2.8e-05
SNR=0.1 Signal 1, Noise 10	Normalized M	2.9876	14.82	102.6
	RE%	0.4133	1.2	2.29
	Hypothesis Value	0.0013	0.0018	0.0017
SNR=1 Signal 10, Noise 10	Normalized M	2.6250	10.63	57.15
	RE%	12.5	29.13	45.57
	Hypothesis Value	0.0383	0.0433	0.0337
SNR=2 Signal 20, Noise 10	Normalized M	2.33	7.96	34.5
	RE%	22.33	46.93	67.14
	Hypothesis Value	0.0684	0.0698	0.0497
SNR=10 Signal 100, Noise 10	Normalized M	1.73	3.9181	10.214
	RE%	42.33	73.88	90.27
	Hypothesis Value	0.1296	0.1099	0.0668

4. Simulation Study

The first simulation validates the normality of the central limit theorem assumed in the proposed hypothesis (12). Only when the normality of the central limit theorem holds can the hypothesis stand. Simulating by iterating 5000 times and each iteration generating 10,000 white noise samples, choose the order $p = 4$ which is essentially the well-known Kurtosis. We choose Kurtosis because it is well accepted in the bearing fault diagnosis, so it well represents the problem. Results seen in Figure 2, the $\frac{\sum |m_i|^p}{n \sigma_m^p}$ centers around the expected value 3 and it is symmetric, which demonstrates the normality property of the empirical Kurtosis values. The hypothesis (12) based on the normality assumption, is thus proper. The spreading of empirical kurtosis values around 3 also suggests that comparing the empirical value of a

moment order directly with the theoretical value is with certain degree of error. Using the hypothesis statistics is necessary to control the error for practical cases.

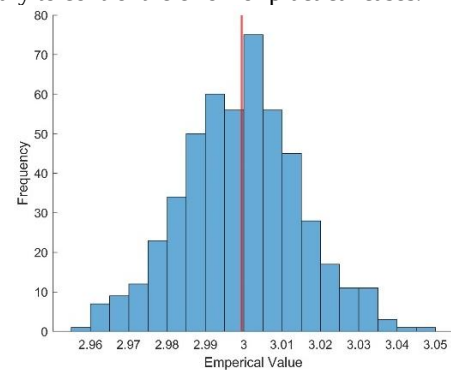


Figure 2. Empirical Kurtosis Values

The next simulation also selects $p = 4$. The signal length chooses sufficiently long as $t = 10,000,000$ with 10,000 data sampled for each signal. Two types of signals are simulated as a pair: pure white noise, and the same white noise plus $a \sin t$. Figure 3 shows the empirical distribution of Kurtosis value for both weak signals (SNR=0.001) and strong signals (SNR = 1). Signal power $\frac{a^2}{2}$ selects as 1 for simplicity. SNR less than 1 suggests noise overwhelming signal, while over 1 suggests signal dominant in the data. It shows for the weak signal (the left of Figure 3 where blue is Kurtosis values for pure white noise, the brown one for mixed signal), the two plots are not visually different, but for the strong signal shown on the right of the Figure 3, the Kurtosis values of the two signals (represented by blue and brown) are separated clearly. The plots of Figure 3 suggest using Kurtosis might not be able to detect weak signals, but for signal of SNR over 1, it should work very well, which is consistent with the conclusion from Section 3.

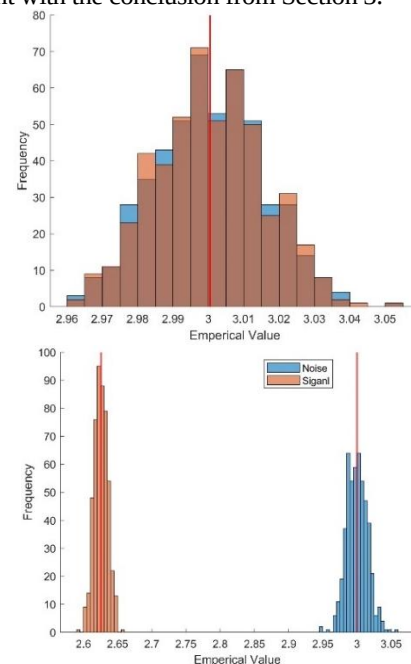


Figure 3. Empirical Kurtosis Values

The visual check using Figure 3 is to check if the Kurtosis, just for demonstration, can be used for detecting defects. It seems for the strong signal, it works, while for weak signal, it is not inadequate. Further simulation runs to check the performance of Kurtosis by using Eq (12), the proposed hypothesis test method. We simulate an iteration of 5000 times for each SNR. The performance is gauged by the accuracy of correctly identifying the signal with defect and the pure noise signal. The signal/noise ratio is setting as 0.001, 0.01, 1 and 10 respectively. Results are shown in Table 3. Using Kurtosis, all signals above SNR 1 can be detected with 100%, either for signal detecting or noise detecting, as shown in the left of Table 3 where the significance level chooses $\alpha = 0.05$. While for the weak signal with SNR less than 1, none signal, corresponding 0% of accuracy, can be detected, meaning completely failed. Setting the significance level to $\alpha = 0.05$, for the weak signal, the performance is just slightly better, as shown in the right of Table 3. Changing the significance level won't help to improve performance. We thus fix the significance level as 0.05.

Table 3. Simulation Results for $p = 4$

S/N Ratio	$\alpha = 0.05$		$\alpha = 0.50$	
	Accuracy of Detecting Defects	Accuracy of Detecting White Noise	Accuracy of Detecting Defects	Accuracy of Detecting White Noise
0.001	0	100	17.8	83
0.01	0	100	17.2	83.2
1	100	100	100	81
10	100	100	100	81.4

For various order p , we check if the performance varies. We select $p = 0.5, 3, 8$ respectively and use the hypothesis proposed in Eq (12). The results are shown in Table 4. It shows changing the order won't change the performance significantly, though slightly difference from each other.

Table 4. Simulation Results for other moment orders

S/N Ratio	Order $p = 0.5$		$p = 3$		$p = 8$	
	Accuracy of Defect Signal	Accuracy of Pure Noise	Accuracy of Defect Signal	Accuracy of Pure Noise	Accuracy of Defect Signal	Accuracy of Pure Noise
0.001	0.04	99.98	0	100	3.04	96.92
0.01	0.02	100	0	100	3.16	96.82
1	100	99.96	100	100	100	96.98
10	100	99.6	100	100	100	97

Next examination uses the p that can maximize the hypothesis value (12). The analytical solution of finding the best order is unobtainable since no implicit expression with Eq (12). We used exhausting approach, albeit computational inefficiently but effectively, to find the

best order p . The Figure 4 plots the hypothesis values changed with the order, where the left is with SNR less than 1, the right is SNR bigger than 1. It is desirable if the hypothesis values from Noise and Defect Signal differ significantly. For the strong defect signal, the difference is significant as shown on the right of Figure 4; while for the weak signal shown on the left of Figure 4, they mix with only slight difference in the tail, which is undesirable. We can conclude that the defect cannot be detected when defect signal is not strong, even using the best moment order.

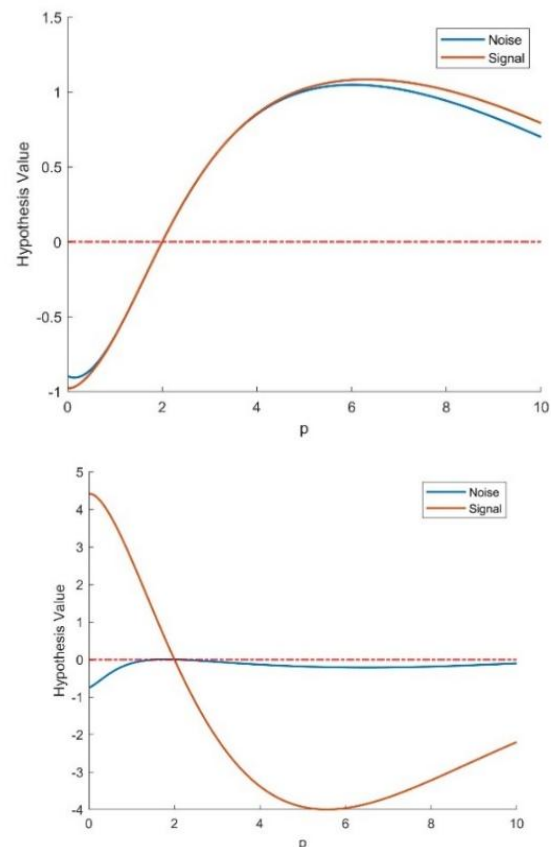


Figure 4. Hypothesis values of various real-valued order

A simulation runs to examine the performance of using the p that maximizes the hypothesis values. Simulation for each group only runs for 500 iterations due to the low computational efficiency of the exhausting method. The accuracies for each SNR are shown in Table 5 where significance level 0.05 and 0.5 is selected.

Table 5. Simulation Results using best order p

S/N Ratio	$\alpha = 0.05$		$\alpha = 0.50$	
	Accuracy of Defect Signal	Accuracy of Pure Noise	Accuracy of Defect Signal	Accuracy of Pure Noise
0.001	5.6	93.8	74.4	26.4
0.01	6.6	92.80	72.2	27.2
1	100	95.6	100	29.4
10	100	94.8	100	28.8

Analogous to the previous simulation results, for SN ratio under 1, the hypothesis still shows insufficient accuracy, as an accuracy below 50% is meaningless. The moment order failed to detect the weak signal with SNR less than 1. Other methods such as physics-based approaches could help improve accuracy. While for the SNR ratio higher than 1 and above, using hypothesis best p , the signal can be detected with 100%, the accuracy for noise signal is lower but still above 94%. But compared with the results in Table 4, there was no improvement, even lower performance found in Table 5. It suggests it is not necessary to maximize the Eq (12) to find the optimal p , as it tends to label the noise as signal erroneously.

In general, from all simulations, it finds the selection of moment order is not necessary, in terms of detecting accuracy, as no order can have significant better performance for weak signal. The applicability of moment method demonstrates its limitations. This simulation considers very ideal context where harmonic signals, other disturbances and other noise are not present. When other disturbances are present, the order moment will be more sensitive to undesired noise, leading to more misjudgments.

5. Conclusion

Statistical moments are a kind of quantity very sensitive to data fluctuation, so that it is used to detect the change of normality of pure noise data in fault diagnosis. This paper finds the exact statistical distribution of the mixture of defect signal and white noise and proves the theoretical distribution clearly differs from the Normal distribution, which is the mathematical foundation of using the moment method to detect weak signal. However, when using the moment method, this paper finds that there is no significant difference among the various moment orders for fault detection. No order shows significantly better performance than the others, always. The initial hypothesis assumes fault diagnosis accuracy would vary significantly with moment orders, which is thus not found its ground. Consequently, finding the optimal moment order is unnecessary. A similar conclusion was reached by Antoni et al. (2019, 2024) [10, 11], where they claimed the universal optimal order does not exist, as the performances of different orders highly depend on the distribution of both healthy and faulty data. This paper finds that even for the same Gaussian healthy data distribution, the optimal order does not exist universally. Additionally, our simulation reveals a limitation of the moment method when the signal is not dominant over the noise. When the defect signal is too weak to surpass the noise, the moment method struggles to distinguish the signal from the noise effectively. The moment method works only when the defect signal becomes sufficiently strong. This implies that the moment method is not capable of detecting defects as early as expected, particularly in the very early stage when the signal is too weak. Our future work will focus on developing new

methods to detect defects at a very early stage, when the defect signal is extremely weak.

Conflict of interest

No conflict of interest has been expressed by the authors.

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